

γ parameter and Solar System constraint in Scalar-Tensor theory with a power law potential and universal scalar/matter coupling

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The effect of a universal scalar/matter coupling are investigated in Scalar-Tensor theories. It is shown that the metric can be put in its standard post-Newtonian form – in contradiction with ‘ γ parameter and Solar System constraint in chameleon-Brans-Dicke theory’, Phys. Rev. D 83, 104019 (2011), 1201.0271, Saaïdi et al. However, assuming the validity of an effective Lagrangian for the matter field, it is pointed out that $1 - \gamma$ could be either positive, null or negative for finite value of ω , depending on the coupling function; while Scalar-Tensor theories without coupling always predict $\gamma < 1$ for finite value of ω .

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I. INTRODUCTION

Scalar-Tensor (ST) theories are known to be good alternative candidates to General Relativity (GR) [1–4]. From the Occam’s razor point of view, it’s the simplest extension in the framework of 4 dimensional space-time theories of gravity. From a theoretical point of view, many other alternative theories turn out to either be re-writable in a ST theory form; or reduce to ST theories, as for theories with extra-dimensions (see for instance [5], for examples on both cases).

Hence it sounds interesting to test the phenomenology predicted by ST theories. However, today’s observations and experiments show that the phenomenology of the actual theory of gravitation is very close to the one of GR [1]. Especially in the weak field limit, as in the Solar System [3]. However, needless to say, increasing accuracy of observations and experiments will allow to put more and more constraints on any alternative theories. In this context, continuing studying alternative theories as well motivated as ST theories is important in order to correctly interpret data, to see any possible deviation from GR.

A lot of work has been done on ST-theories. However, one of the latest stream in that field is to look after non-minimal coupling of the scalar field with the material part of the Lagrangian. From an extra-dimensional point of view, there is no reason not to consider this possibility – one just has to take a look at the 4-dimensional Lagrangian obtained after compactification of the fifth dimension in a five dimension GR to figure this out [6]. Besides, some toy models suggested by String theories advocate that the coupling between the scalar field and the material fields could be driven toward a weak coupling (but non-null) during cosmological evolution [7, 8], hence giving a reason of the apparent weak coupling that would not involve fine tuning. On the other hand, the possible observation of the variation of the fine structure constant in both time [9] and space [10, 11] may be an indication that there is a coupling between some material fields and a new scalar field (see for instance [12, 13] and references therein). Therefore, the generalization of the usual Brans-Dicke-like ST theory to the case where such a coupling occurs seems very interesting since it generically produces such a space-time dependency of the fundamental coupling constants. Moreover, if one wants to construct an effective theory of fundamental interactions, some requirements such as gauge and diffeomorphism invariances essentially single out a particular set of theories which turns out to be Brans-Dicke-like ST theories with scalar/matter coupling [14].

One of the simplest model of such a coupling is given by a field that couples universally to all the material field through a function in factor of the material part of the Lagrangian¹. Such a field has been dubbed *chameleon* by [15, 16], even though the chameleon field originally refers to the *thin shell mechanism* [17] that appears in some models where the scalar/matter coupling appears through conformal factors in front of the metric entering in the material part of the Lagrangian [17–19]. Since it is easily seen that there is no conformal transformation that goes from the universal scalar/mater coupling action to the chameleon action, and since there is no proof we are aware of that the scalar-field we consider would generically behave as a *chameleon*, we won’t dubb our scalar field *chameleon*.

¹ Note that it differs from Damour and Polyakov toy model [7] by the fact that the universal coupling function is a factor of the material part of the Lagrangian only.

Especially, the Post-Newtonian (PN) limit of the Brans-Dicke theory for a universal scalar/matter coupling has been worked out by Saaïdi et al. [16] in the case of an inverse power law potential. In a recent comment [20], we argued about several mistakes made in Saaïdi et al. paper. In the current paper, we derive what we think is the actual post-Newtonian behavior of ST theories with universal scalar/matter coupling.

In section II, we derive the equations of motion coming from the considered action. Then, in section III we study the first Post-Newtonian perturbative order of such theories. Section IV then gives the trivial generalization to the full Scalar-Tensor case. Finally, section V is about some possible links of the theory considered here to extra-dimensional theories.

Throughout the paper, the notation conventions of the International Astronomical Union are used [21].

II. EQUATIONS OF MOTION

The action describing Brans-Dicke theory with a universal scalar/matter coupling can be written as follows:

$$S = \int d^4x \sqrt{-g} \left(\Phi R - \frac{\omega}{\Phi} (\partial_\sigma \Phi)^2 - V(\Phi) + 2f(\Phi) \mathcal{L}_m(g_{\mu\nu}, \Psi) \right), \quad (1)$$

where g is the metric determinant, R is the Ricci scalar constructed from the *physical* metric $g_{\mu\nu}$ ², \mathcal{L}_m is the material Lagrangian, Ψ represents the non-gravitational fields and $V(\Phi) \propto \Phi^{-n}$, with $n \in \mathbb{R}$. From this action, and defining

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g} \mathcal{L}_m)}{\delta g^{\mu\nu}}, \quad (2)$$

one gets the following equations of motion:

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R &= \frac{f(\Phi)}{\Phi} T_{\mu\nu} + \frac{\omega}{\Phi^2} (\partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2} g_{\mu\nu} (\partial_\alpha \Phi)^2) \\ &+ \frac{1}{\Phi} [\nabla_\mu \nabla_\nu - g_{\mu\nu} \square] \Phi - g_{\mu\nu} \frac{V(\Phi)}{2\Phi}, \end{aligned} \quad (3)$$

and

$$\frac{2\omega + 3}{\Phi} \square \Phi = \frac{f(\Phi)}{\Phi} T - 2f_{,\Phi}(\Phi) \mathcal{L}_m + V_{,\Phi}(\Phi) - 2 \frac{V(\Phi)}{\Phi}. \quad (4)$$

III. THE POST-NEWTONIAN DEVELOPMENT

Lets write the perturbations of the fields as follow:

$$\Phi = \Phi_0 + c^{-2} \varphi \quad (5)$$

$$g_{\mu\nu} = \eta_{\mu\nu} + c^{-2} h_{\mu\nu} + O(c^{-3}), \quad (6)$$

where $\eta_{\mu\nu}$ is the metric of Minkowski and Φ_0 is constant background field. It has to be stressed out that equation (6) is always valid in a local enough region of space-time³. Therefore, we are choosing not to restrict ourselves to the asymptotic flatness case. However, observations tell us that $g_{00}g_{ij} = -\delta_{ij}$ + unobserved quantities (so far), where $g_{00} \sim -1 + c^{-2} \{2G_{eff}M/r\} + O(c^{-4})$. Therefore, because of equations (3-4), the quantities $V(\Phi_0)/\Phi_0$ and $V_{,\Phi}|_{\Phi_0}$ are at best of order $8\pi G_{eff}T^{00}/c^4$, where G_{eff} is the effective (or local) gravitational constant as measured at low redshifts. In other words, the potential terms are at best first order terms; while Saaïdi et al. assumed that they were zeroth order terms in [16]⁴. Also, because we don't expect the potentials to be related to the particular solar system's energy density, it would be a suspicious coincidence that those terms would be precisely of order $8\pi G_{eff}T^{00}/c^4$ and

² In our model, we assume that $g_{\mu\nu}$ is the *physical* metric, in the sens that it is the one that describes actual time and length as measured by clocks and rods in our experiments [1].

³ The size L of such a region can be linked in some situations to the effective cosmological constant λ_{eff} through $L \ll \sqrt{1/\lambda_{eff}}$ [22].

⁴ As a result, they obtained an asymptotically flat solution that is however incompatible with their field equation where $V(\Phi)$ and $V_{,\Phi}(\Phi)$ wouldn't be asymptotically null.

not arbitrarily bellow. Indeed, a priori they could have any value that is not constrained by experiments (ie. roughly speaking, any value far less than $8\pi G_{eff} T^{00}/c^4 \sim 10^{-23} m^{-2}$ for Solar system objects). Therefore when dealing with Scalar-Tensor theories, while potentials can play an important role at cosmological scales, one either not considers potential at the Solar System scale [1, 2, 23, 24]; or considers only fine-tuned potentials where $V(\Phi_0) = V_{,\Phi}(\Phi_0) = 0$, while $V_{,,\Phi}(\Phi_0) = m^2$ [25–27])⁵. Indeed, if $V(\Phi_0) \neq 0$, equation (3) implies

$$\frac{V(\Phi_0)}{\Phi_0} = O(\lambda_{eff}), \quad (7)$$

where λ_{eff} is the effective (or local) cosmological constant as measured for "low-enough" redshifts. Observations tell us that $\lambda_{eff} \sim 10^{-52} m^{-2}$ while, for Solar system objects, one has $8\pi G_{eff} T^{00}/c^4 \sim 10^{-23} m^{-2}$. Besides, one has

$$V_{,\Phi}(\Phi_0) = -n \frac{V(\Phi_0)}{\Phi_0} + O(c^{-2} n \lambda_{eff}). \quad (8)$$

Moreover, equation (3) leads to $f(\Phi_0)/\Phi_0 \sim 8\pi G_{eff}/c^4$. Hence, unless $|n| \gtrsim 10^{29}$, $V_{,\Phi}$ can also be considered as a negligible quantity at the first perturbative order. (Of course, the discussion actually depends on the scale considered, as one expects from the relaxation of the asymptotic flatness assumption – see appendix A for a more detailed discussion). Hence, one writes

$$V_{,\Phi}(\Phi_0) = O(n \lambda_{eff}). \quad (9)$$

Now, assuming that at the lowest perturbative order one has $\mathcal{L}_m = -c^2 \rho + O(c^0) = T + O(c^0)$ [28–32], equations (3) and (4) can be re-written at the first perturbative order as the following:

$$R^{\mu\nu} = \frac{f(\Phi_0)}{\Phi_0} \left(T^{\mu\nu} - \frac{1}{2} g^{\mu\nu} T \right) + \frac{1}{\Phi_0} \left(\partial^\mu \partial^\nu + \frac{1}{2} g^{\mu\nu} \Delta \right) \Phi + O(c^{-4}, \lambda_{eff}), \quad (10)$$

$$\frac{2\omega + 3}{\Phi_0} \square \Phi = (1 + \Delta) \frac{f(\Phi_0)}{\Phi_0} T + O(c^{-4}, \lambda_{eff}, n \lambda_{eff}), \quad (11)$$

where

$$\Delta \equiv -\zeta \Phi_0 \ln_{,\Phi}(f(\Phi))|_{\Phi_0}, \quad (12)$$

ζ being equal to 2 in our case. (The case treated by [16] would correspond to $\zeta = 1/2$). From there, we can notice three possible cases :

- $\left| \frac{1+\Delta}{2\omega+3} \right| \gtrsim 1/2$: where the main source of the curvature is the second part of the rhs. of (10), which contains the scalar field terms. Therefore we dubb this case, the *strong coupling* regime.
- $\Delta = -1 (\Rightarrow \epsilon = 0) \Rightarrow f(\Phi) = \alpha \Phi^{1/\zeta} + n(\Phi)$, where α is a constant, $n(\Phi_0)$ and $n_{,\Phi}(\Phi_0)$ are small enough to be neglected in the field equations at the first perturbative order and $\zeta \neq 0$. In that case, the scalar-field is not coupled to matter at the first order and the PN limit of GR is recovered. Otherwise, one should notice that for $f(\Phi) = \alpha \Phi^{1/\zeta}$, GR equations are totally recovered for the field solution $\Phi = \Phi_0$.
- $0 < \left| \frac{1+\Delta}{2\omega+3} \right| < 1/2$: where the main source of the curvature is the first part of the rhs. of (10). Therefore we dubb this case, the *weak coupling* regime.

As one shall notice later on, it turns out that if one wants to have $\gamma \sim 1$, one needs to either be in the weak coupling regime or in the non-coupling one.

Since we are interested in the γ parameter only, and in order to simplify the discussion, we won't discuss the c^{-3}

⁵ Note that a priori, there is no reason why – following $V(\Phi_0)$ and $V_{,\Phi}(\Phi_0) - V_{,,\Phi}(\Phi_0)$ shouldn't be arbitrarily small. Therefore, in the following we will refer to theories with $V_{,,\Phi}(\Phi_0) = m^2$ as *locally massive ST*.

metric term. Therefore, we re-write the field equations at the 1-PN/RM level ⁶:

$$R^{\mu\nu} = \frac{f(\Phi_0)}{\Phi_0} \left(T^{\mu\nu} - \frac{1}{2} g^{\mu\nu} T \right) + \frac{c^{-2}}{\Phi_0} \left(\partial^\mu \partial^\nu + \frac{1}{2} g^{\mu\nu} \Delta \right) \varphi + O(c^{-3}, \lambda_{eff}, n \lambda_{eff}), \quad (13)$$

$$\frac{2\omega + 3}{\Phi_0} \Delta \Phi = (1 + \Delta) \frac{f(\Phi_0)}{\Phi_0} T + O(c^{-3}, \lambda_{eff}, n \lambda_{eff}). \quad (14)$$

(In the following, for simplicity we shall write $O(\lambda_{eff}, n \lambda_{eff}) = O(\lambda_{eff})$.) These equations correspond to the equations of a non-massive BD theory – with universal scalar/matter coupling – at the first PN/RM order. Defining

$$\sigma \equiv T^{00}/c^2 + O(c^{-2}), \quad (15)$$

$$G_{eff} \equiv \left(1 + \frac{1 + \Delta}{2\omega + 3} \right) \frac{c^4}{8\pi} \frac{f(\Phi_0)}{\Phi_0}, \quad (16)$$

$$\gamma \equiv \frac{2\omega + 2 - \Delta}{2\omega + 4 + \Delta}, \quad (17)$$

the previous equations can be re-written as follows:

$$R^{00} = c^{-2} \{ 4\pi G_{eff} \sigma \} + O(c^{-3}, \lambda_{eff}), \quad (18)$$

$$R^{ij} = c^{-2} \left\{ -\delta_{ij} \gamma 4\pi G_{eff} \sigma + \frac{1}{\Phi_0} \partial_i \partial_j \varphi \right\} + O(c^{-3}, \lambda_{eff}) \quad (19)$$

$$\frac{1}{\Phi_0} \Delta \varphi = -\frac{2 + 2\Delta}{2\omega + 4 + \Delta} 4\pi G_{eff} \sigma + O(c^{-1}, c^2 \lambda_{eff}). \quad (20)$$

It is then straightforward to show that the metric solution can be put under the following standard PN/RM form

$$g_{00} = -1 + c^{-2} \frac{2w}{c^2} + O(c^{-3}, \lambda_{eff}), \quad (21)$$

$$g_{0i} = O(c^{-3}, \lambda_{eff}), \quad (22)$$

$$g_{ij} = \delta_{ij} \left(1 + c^{-2} \frac{2\gamma w}{c^2} \right) + O(c^{-3}, \lambda_{eff}), \quad (23)$$

where γ is indeed a constant given by (17), and where w satisfies the equation of Newton at the first perturbative order :

$$\Delta w = -4\pi G_{eff} \sigma + O(c^{-1}, c^2 \lambda_{eff}). \quad (24)$$

The important fact to notice is that, depending on the value of Δ (and thus depending on the coupling function), $1 - \gamma$ could be either positive, null ⁷ or negative; while STT without coupling predict a positive value for finite value of ω .

A. Remark on the current constraints coming from propagation of light observations

As shown in appendix B, the geometric optic limit of the modified Maxwell equations leads to the usual geodesic equation for the propagation of light. Hence, from (17), one gets that the weak coupling is constrained by observations involving propagation of light as follows:

$$\left| \frac{1 + \Delta}{2\omega + 3} \right| \lesssim \frac{|1 - \gamma_{obs}|}{2}, \quad (25)$$

where γ_{obs} is the value given by current observational constraints on the PN parameter γ .

⁶ PN/RM stands for Post Newtonian/Relativistic Motion. It means that the development of the Post-Newtonian metric is developed to the order that has to be taken into account when dealing with test particles with relativistic velocities only. On the contrary, PN/SM stands for Post-Newtonian/Slow Motion. It means that the development of the Post-Newtonian metric is developed to the order that has to be taken into account when dealing with test particles with non-relativistic velocities only [23, 33].

⁷ The author has recently became aware of the paper of Moffat and Toth [34] in which they explored such a possibility in order to argue the possible solar system viability of Modified Gravity Theory (MOG) [35, 36].

IV. GENERALIZATION TO THE SCALAR-TENSOR CASE

The generalization of the action (1) to the general scalar-tensor case results in the following modifications of the field equations (3)-(4)

$$\begin{aligned} & \Phi(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R) \\ &= f(\Phi)T_{\mu\nu} + \frac{\omega(\Phi)}{\Phi}(\partial_\mu\Phi\partial_\nu\Phi - \frac{1}{2}g_{\mu\nu}(\partial_\sigma\Phi)^2) \\ &+ [\nabla_\mu\nabla_\nu - g_{\mu\nu}\square]\Phi - g_{\mu\nu}\frac{V(\Phi)}{2}, \end{aligned} \quad (26)$$

and

$$\begin{aligned} (2\omega(\Phi) + 3)\square\Phi &= f(\Phi)T - 2\Phi f_{,\Phi}(\Phi)\mathcal{L}_m \\ &+ (\Phi V_{,\Phi}(\Phi) - 2V(\Phi)) - \omega_{,\Phi}(\partial_\sigma\Phi)^2. \end{aligned} \quad (27)$$

The 1-PN/RM limit of equations (26-27) is the same than for the simple BD case considered previously. It is not surprising that the field equations at the 1-PN/RM order are the same in both the Scalar-Tensor and Brans-Dicke cases. Indeed, the changes from equations (3)-(4) and (26)-(27) arise in factor of $(\partial_\sigma\Phi)^2$ -like terms that are of order $O(c^{-4})$. Thus, those terms will impact the calculation at the 1PN/SM or 2PN/RM level only [23].

Therefore, our result (21)-(24) holds for the Scalar-Tensor case (with ω replaced by $\omega(\Phi_0)$).

Otherwise, using the Einstein representation gives the same results as one can see in appendix D.

V. REMARK REGARDING THEORIES WITH EXTRA DIMENSIONS

In most cases, compactified extra dimensions imply ω to be of the order of -1 [5, 37]. Hence, in order to pass solar system tests that give $\omega \gtrsim 10^4$ for non-massive scalar-fields [26], extra-dimensional theories have to imply a sufficient mass for the effective scalar field. Indeed, a very massive scalar-field has a frozen spatial dynamics on solar system scales, such that every value of ω satisfy solar system observations [26] ($|1 - \gamma| \lesssim 10^{-4}$ [38]). However, in [39] is described a mechanism allowing to get rid of this condition. But as we can see, the universal scalar/matter coupling allows to do the same. Indeed, as long as $\Delta = -1 + |\mu|$, with μ small enough, the theory could pass the γ -parameter solar system tests, no matter the value of ω . However, at the phenomenological level considered here, it requires a fine-tuning of the theory. Besides, it is worth stressing that those theories are also constrained by the tests of the equivalence principle. For instance, among other violations of the Equivalence Principle – such as the variation of the fundamental *constants* – there is an energy transfer between material and scalar fields such that the conservation equation writes:

$$\nabla_\sigma [f(\Phi)T^{\mu\sigma}] = \mathcal{L}_m f_{,\phi}(\Phi)\partial^\mu\Phi. \quad (28)$$

However, such an energy transfer can actually have some virtue, as shown for general material to scalar-field energy transfers in [40] or more specifically for the theory considered here in [15]. Besides, one could invoke the so-called *Least Coupling Principle* in order to explain the smallness of the current coupling [7, 8]. Indeed, it has been argued that in some cases (see the following subsection as well as appendix C), the scalar-field is naturally driven toward a decoupling to matter through the cosmic evolution. In that case, one could have $\partial_\alpha\Phi|_{\Phi_0} \sim 0$; while Δ could still be finite (see next subsection).

A. Example: Damour&Polyakov Toy Model

From the observation in [8] that a kind of universality between the coupling functions leads to a strong attracting mechanism – realising the so-called *Principle of Least Coupling* – Damour and Polyakov proposed a toy model in [7] – suggested by an effective string massless modes action in String theories – with one universal coupling function $B(\Psi)$:

$$S = \int d^4x \sqrt{-g} B(\Psi) \left[R + 4\square\Psi - 4(\partial_\sigma\Psi)^2 + \alpha\mathcal{L}_m \right]. \quad (29)$$

Admitting that $B(\Psi) \equiv \Phi$ is invertible (at least locally) and that the inversion around Ψ_0 writes $\Psi = A(\Phi)$ – along with the fact that divergences don’t contribute to the equations of motion – one shows that the previous action reduces to ⁸:

$$S = \int d^4x \sqrt{-g} \left(\Phi R - \frac{\omega(\Phi)}{\Phi} (\partial_\sigma \Phi)^2 + \alpha \Phi \mathcal{L}_m \right), \quad (30)$$

with

$$\omega(\Phi) \equiv 4\Phi A_{,\Phi}(\Phi) [1 + A_{,\Phi}(\Phi)]. \quad (31)$$

It means that their toy model locally corresponds to $f(\Phi) = \alpha\Phi/2$. Now, if one takes the case where $\zeta = 2$ – as advocates by section II – then one finds $\Delta = -2$. Thus, thanks to equation (17), one deduces $\gamma > 1$. (It is important to notice that one finds $\gamma > 1$ as well when considering their original action without the inversion considered here. A generalization to any parametrization can be found in the appendix C.).

VI. CONCLUSION

First, it is shown in this paper that Brans-Dicke theories with a power law potential and a universal scalar/matter coupling reduces to the usual standard PN metric at the 1PN/RM order, as soon as one takes actual numerical values into account. Moreover, it is emphasized that the result holds when the BD theory is generalized to the ST theory. In both cases, the only difference regarding the metric with the case considered in Will’s book for instance [2], is that the coefficient γ is modified because of the coupling of the scalar field with matter:

$$\gamma \equiv \frac{2\omega + 2 - \Delta}{2\omega + 4 + \Delta}, \quad (32)$$

$$\Delta \equiv -\zeta \Phi_0 \ln_{,\Phi} f(\Phi)|_{\Phi_0} \quad (33)$$

However, conversely to STT without coupling, it is shown that the $1 - \gamma$ can be either positive, null or negative, depending on the coupling function $f(\Phi)$. The knowledge of this behavior is crucial in order not to wrongly rule out ST theories if the post-Newtonian parameter γ was measured to be more than one.

Second, it is argued that such a scalar-field, if generated by compactification of extra dimensions, could allow theories with compactified extra dimensions to pass solar system tests, even if they don’t lead to a mass for the scalar field in the reduced 4-dimension space-time.

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⁸ Here, the only goal of the inversion is to write the equations in the form considered in the current paper. Using the original Lagrangian of Damour and Polyakov doesn’t change the main result of this sub-section – namely $\gamma > 1$ in their model.

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Appendix A: λ_{eff} and the scale of its influence

While the matter source term in the field equations is bounded, the “ λ ” source terms are not and hence, their influence grow with the increasing of the scale considered. Mathematically speaking, it means that while the former leads to “external” spherical harmonic solutions (ie. $w \propto 1/r^L$), the latter leads to “internal” spherical harmonic solutions (ie. $w \propto r^L$). Thus, the discussion about orders of magnitude in section III has to be specified – although the huge difference between the value of $4\pi G_{eff} T^{00}/c^4$ and λ_{eff} don’t leave many doubts regarding the smallness of the λ contribution at the PN level.

The 1-PN/RM field equations for which the potential and its derivative have not been neglected write:

$$c^2 R^{00} = \frac{c^4 f(\Phi_0)}{2\Phi_0} \sigma - \frac{\Delta \varphi}{2\Phi_0} - c^2 \frac{V(\Phi_0)}{2\Phi_0} + O(c^{-2}), \quad (A1)$$

$$\begin{aligned} \frac{\Delta \varphi}{\Phi_0} &= \frac{1 + \Delta}{2\omega + 3} \frac{f(\Phi_0)}{\Phi_0} \sigma + \frac{c^2}{2\omega + 3} \left\{ V_{,\Phi}(\Phi_0) - \frac{2V(\Phi_0)}{\Phi_0} \right\} \\ &+ O(c^{-2}). \end{aligned} \quad (A2)$$

Combining those two equations, one gets the following solution for the gravitational potential at the Newtonian level:

$$w = G_{eff} \int \frac{\sigma(t, \vec{x}')}{|\vec{x} - \vec{x}'|} d^3 x' + \frac{\lambda^*}{6} r^2 + O(c^{-2}), \quad (A3)$$

where

$$\lambda^* \equiv c^2 \left\{ \frac{2\omega + 1}{4\omega + 6} \frac{V(\Phi_0)}{\Phi_0} + \frac{V_{,\Phi}(\Phi_0)}{4\omega + 6} \right\}. \quad (A4)$$

One can note that this simple result 1/ coincides with GR case – with a non-null cosmological constant – at the Newtonian order (with λ replaced by λ^*), 2/ is quite different from the result obtained by [16] and 3/ is in accordance with the relaxation of the asymptotic flatness assumption in section III.

Since we saw that in this class of model, ω is not trivially constrained by observations, the second term of the r.h.s. is not necessarily small compared to the first term, and thus should not be neglected a priori. However, under the class of potentials considered here, and assuming that $|n|$ is small enough, we shall consider that $V_{,\Phi}(\Phi_0) = O(V(\Phi_0)/\Phi_0)$, such that we indeed have $c^{-2}\lambda^* = O(\lambda_{eff}) \sim 10^{-52} m^{-2}$. Under those assumptions, if we require that the λ -term stays N orders of magnitude below the *Newtonian-term*, it leads to the following requirement:

$$r < \left(\frac{6G_{eff}M}{10^N c^2 \lambda_{eff}} \right)^{1/3}. \quad (A5)$$

For the solar system, a safety of 6 orders of magnitude implies that the scale of the experiment has to be less than 1 parsec (~ 3 light-years). From this, we conclude that it is safe enough to neglect the potential in the field equations, if the potential is linked to the value of the locally measured cosmological constant as discussed in section III.

Also, one should notice that the consequences of a gravitational potential with a diverging r^2 -term as in (A3) have been thoroughly studied and compared to data (see for instance [41] and references therein).

Appendix B: The Geometric Optic limit

Lets consider the geometric optic limit of Electromagnetism in the theory studied here. The electromagnetic field amplitude considered being extremely weak – a laser of a few Watts for instance – the electromagnetic field won't affect the field equations previously considered (ie. photons are considered as test particles). However, the Maxwell equation in vacuum is modified by the scalar field in the following way:

$$\nabla_\sigma (f(\Phi) F^{\mu\sigma}) = 0. \quad (B1)$$

Using the Lorenz Gauge ($\nabla_\sigma A^\sigma = 0$), along with equation (20), one puts this equation under the 1-PN/RM following form:

$$-\square A^\mu + g^{\mu\epsilon} R_{\gamma\epsilon} A^\gamma + \varkappa (\nabla^\mu A^\sigma - \nabla^\sigma A^\mu) \partial_\sigma w = O(c^{-3}), \quad (B2)$$

where

$$\varkappa \equiv c^{-2} \Phi_0 \frac{f_{,\Phi}}{f} \Big|_{\Phi_0} \frac{2 + 2\Delta}{2\omega + 4 + \Delta}. \quad (B3)$$

Following the analysis made in [42], we expand the 4-vector potential as follows:

$$A^\mu = \Re \left\{ (a^\mu + \epsilon b^\mu + O(\epsilon^2)) \exp^{i\theta/\epsilon} \right\}, \quad (B4)$$

The two first leading orders of equation (B2) respectively give:

$$k_\sigma k^\sigma = 0, \quad (B5)$$

where $k_\sigma \equiv \partial_\sigma \theta$, and

$$a^\mu \nabla_\sigma k^\sigma + 2k^\sigma \nabla_\sigma a^\mu + \varkappa (k^\mu a^\sigma - k^\sigma a^\mu) \partial_\sigma w = 0. \quad (B6)$$

Remembering that the Lorenz Gauge condition gives $k_\sigma a^\sigma = 0$ at the leading order, one gets:

$$k^\sigma \nabla_\sigma k^\mu = 0. \quad (B7)$$

This equation is the usual geodesic equation, showing that the presence of the scalar-field won't affect light ray trajectories at the geometric optic approximation. However, defining $a^\mu = af^\mu$, the propagation equation for the scalar amplitude (a) as well as the propagation equation for the polarization vector (f^μ) are modified:

$$k^\sigma \nabla_\sigma a = -\frac{a}{2} \nabla_\sigma k^\sigma + \frac{\varkappa}{2} a k^\sigma \partial_\sigma w, \quad (\text{B8})$$

$$k^\sigma \nabla_\sigma f^\mu = -\frac{\varkappa}{2} k^\mu f^\sigma \partial_\sigma w. \quad (\text{B9})$$

From there follows that the conservation law of "photon number" is modified:

$$\nabla_\sigma (k^\sigma a^2) = \varkappa a^2 k^\sigma \partial_\sigma w. \quad (\text{B10})$$

One notes that the last three equations give alternative ways to put constraints on those theories. Those ways should be investigated using the relevant literature.

Otherwise, one should notice that $\Delta = -1$ implies $\varkappa = 0$ in addition to $\gamma = 1$.

Appendix C: General parametrization of the scalar field

If, instead of action (1), one starts with the following general action:

$$S = \int d^4x \sqrt{-g} \left\{ F(\Phi) R - Z(\Phi) (\partial_\sigma \Phi)^2 + f(\Phi) \mathcal{L}_m(g_{\mu\nu}, \Psi) \right\}, \quad (\text{C1})$$

then the γ parameter re-writes:

$$\gamma = \frac{2ZF + (2 - \Delta)(F_{,\Phi})^2}{2ZF + (4 + \Delta)(F_{,\Phi})^2} \Big|_{\Phi_0}, \quad (\text{C2})$$

with

$$\Delta \equiv -2 \frac{F}{F_{,\Phi}} \Big|_{\Phi_0} \frac{f_{,\Phi}}{f} \Big|_{\Phi_0}. \quad (\text{C3})$$

One can note that, as emphasized in section V A, having $F = f$ as in the Damour and Polyakov's case implies $\Delta = -2$ and therefore $\gamma > 1$.

Otherwise, let us remark that $f = \sqrt{F}$ leads to $\gamma = 1$ (as well as $\varkappa = 0$, cf. appendix B) regardless the value of the kinetic function Z . Besides, one should notice that f and F would then share the same local minimums. Hence, according to Damour and Polyakov's study [8], such a theory is driven toward a weak scalar/matter coupling through cosmic evolution. Therefore, such theories (ie. for any Z , and any F with a local minimum Φ_m) can be in perfect accordance with the solar system phenomenology, as it is currently known.

Appendix D: Using the Einstein representation

The result presented in this paper does not depend on the representation used to do the calculations as long as one has fixed the *physical* metric – ie. as long as one has explicitly fixed in the theory, which metric characterizes the proper time (τ) in actual experiments ($d\tau^2 = g_{\alpha\beta} dx^\alpha dx^\beta$, where $g_{\alpha\beta}$ is the so-called *physical* metric) [1]. However, it is always interesting to re-derive the calculations in the Einstein representation in order to check the results obtained while using the Jordan representation only. Let's consider the massless case, the action writes in the Jordan and Einstein representation respectively as follows:

$$S = \int d^4x \sqrt{-g} \left(\Phi R - \frac{\omega(\Phi)}{\Phi} g^{\alpha\beta} \partial_\alpha \Phi \partial_\beta \Phi \right) + S_m, \quad (\text{D1})$$

$$= \int d^4x \sqrt{-\tilde{g}} \left(\tilde{R} - \left(\omega(\Phi(\varphi)) + \frac{3}{2} \right) \tilde{g}^{\alpha\beta} \partial_\alpha \varphi \partial_\beta \varphi \right) + S_m, \quad (\text{D2})$$

where $g^{\alpha\beta} \equiv \Phi \tilde{g}^{\alpha\beta}$, $\sqrt{-g} = \Phi^{-2} \sqrt{-\tilde{g}}$ and $\varphi \equiv \ln \Phi$. By definition, the material part of the action (S_m) writes:

$$S_m = \int d^4x \sqrt{-g} \, 2f(\Phi) \mathcal{L}_m, \quad (\text{D3})$$

$$= \int d^4x \sqrt{-\tilde{g}} \, 2f(\Phi(\varphi)) \tilde{\mathcal{L}}_m. \quad (\text{D4})$$

Therefore, by definition, one has $\tilde{\mathcal{L}}_m = \Phi^{-2} \mathcal{L}_m$. The variation of equation (D3) for relevant fields leads to:

$$\delta S_m = \int d^4x \sqrt{-g} \left(-f(\Phi) T_{\alpha\beta} \delta g^{\alpha\beta} + 2f_{,\Phi}(\Phi) \mathcal{L}_m \delta\Phi \right). \quad (\text{D5})$$

Now, since one has $g^{\alpha\beta} \equiv \Phi \tilde{g}^{\alpha\beta}$, the variation of the *physical* metric gives

$$\delta g^{\alpha\beta} = \tilde{g}^{\alpha\beta} \delta\Phi + \Phi \delta\tilde{g}^{\alpha\beta}. \quad (\text{D6})$$

Therefore, equation (D5) writes:

$$\begin{aligned} \delta S_m = & \int d^4x \sqrt{-g} \left(-\Phi f(\Phi) T_{\alpha\beta} \delta\tilde{g}^{\alpha\beta} \right. \\ & \left. + [-f(\Phi) \tilde{g}^{\alpha\beta} T_{\alpha\beta} + 2f_{,\Phi}(\Phi) \mathcal{L}_m] \delta\Phi \right). \end{aligned} \quad (\text{D7})$$

Now, using $T_{\alpha\beta} = \Phi \tilde{T}_{\alpha\beta}$, $\tilde{T} \equiv \tilde{g}^{\alpha\beta} \tilde{T}_{\alpha\beta}$, $\delta\varphi = \delta\Phi/\Phi$ and $\Phi f_{,\Phi}(\Phi) = f_{,\varphi}(\Phi(\varphi))$, one gets:

$$\begin{aligned} \delta S_m = & \int d^4x \sqrt{-\tilde{g}} \left(-f(\Phi(\varphi)) \tilde{T}_{\alpha\beta} \delta\tilde{g}^{\alpha\beta} \right. \\ & \left. - \left[1 - 2 \frac{f_{,\varphi}(\Phi(\varphi))}{f(\Phi(\varphi))} \frac{\tilde{\mathcal{L}}_m}{\tilde{T}} \right] f(\Phi(\varphi)) \tilde{T} \delta\varphi \right). \end{aligned} \quad (\text{D8})$$

However, since $\tilde{\mathcal{L}}_m = \Phi^{-2} \mathcal{L}_m$ and $\tilde{T} = \Phi^{-2} T$, then $\tilde{\mathcal{L}}_m/\tilde{T}$ reduces to $\mathcal{L}_m/T = 1 + O(c^{-2})$ [32]. Therefore, for $\Delta = -1$, one derives that $\varphi = C + O(c^{-4})$, where C is an arbitrary constant. But in the Einstein representation, the metric satisfies the Strong Spatial Isotropy Condition [43] (ie. $\gamma = 1$ for the metric in the Einstein representation). Indeed, the space-space part of Einstein tensor in the Einstein representation is of order c^{-4} ($\tilde{R}^{ij} - 1/2 \tilde{g}^{ij} \tilde{R} = O(c^{-4})$) as one can easily check from the field equations obtained from the action (D2) (see discussions in [23] and [43]).

Therefore, when $\Delta = -1$, the conformal transformation $g_{\alpha\beta} = e^{-\varphi} \tilde{g}_{\alpha\beta} = e^{-C} \tilde{g}_{\alpha\beta} + O(c^{-4})$ corresponds to a simple redefinition of the metric's units at the 1.5PN/RM level, and one has $\gamma = 1$ in the metric of the Jordan representation as well.

In conclusion, one indeed gets the same result whether one derives everything in the Jordan representation or uses the Einstein representation as an intermediate step.

γ parameter and Solar System constraint in Scalar-Tensor theory with a power law potential and universal scalar/matter coupling

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The effect of a universal scalar/matter coupling are investigated in Scalar-Tensor theories. It is shown that the metric can be put in its standard post-Newtonian form – in contradiction with ‘ γ parameter and Solar System constraint in chameleon-Brans-Dicke theory’, Phys. Rev. D 83, 104019 (2011), 1201.0271, Saaïdi et al. However, assuming the validity of an effective Lagrangian for the matter field, it is pointed out that $1 - \gamma$ could be either positive, null or negative for finite value of ω , depending on the coupling function; while Scalar-Tensor theories without coupling always predict $\gamma < 1$ for finite value of ω .

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I. INTRODUCTION

Scalar-Tensor (ST) theories are known to be good alternative candidates to General Relativity (GR) [1–4]. From the Occam’s razor point of view, it’s the simplest extension in the framework of 4 dimensional space-time theories of gravity. From a theoretical point of view, many other alternative theories turn out to either be re-writable in a ST theory form; or reduce to ST theories, as for theories with extra-dimensions (see for instance [5], for examples on both cases).

Hence it sounds interesting to test the phenomenology predicted by ST theories. However, today’s observations and experiments show that the phenomenology of the actual theory of gravitation is very close to the one of GR [1]. Especially in the weak field limit, as in the Solar System [3]. However, needless to say, increasing accuracy of observations and experiments will allow to put more and more constraints on any alternative theories. In this context, continuing studying alternative theories as well motivated as ST theories is important in order to correctly interpret data, to see any possible deviation from GR.

A lot of work has been done on ST-theories. However, one of the latest stream in that field is to look after non-minimal coupling of the scalar field with the material part of the Lagrangian. From an extra-dimensional point of view, there is no reason not to consider this possibility – one just has to take a look at the 4-dimensional Lagrangian obtained after compactification of the fifth dimension in a five dimension GR to figure this out [6]. Besides, some toy models suggested by String theories advocate that the coupling between the scalar field and the material fields could be driven toward a weak coupling (but non-null) during cosmological evolution [7, 8], hence giving a reason of the apparent weak coupling that would not involve fine tuning. On the other hand, the possible observation of the variation of the fine structure constant in both time [9] and space [10, 11] may be an indication that there is a coupling between some material fields and a new scalar field (see for instance [12, 13] and references therein). Therefore, the generalization of the usual Brans-Dicke-like ST theory to the case where such a coupling occurs seems very interesting since it generically produces such a space-time dependency of the fundamental coupling constants. Moreover, if one wants to construct an effective theory of fundamental interactions, some requirements such as gauge and diffeomorphism invariances essentially single out a particular set of theories which turns out to be Brans-Dicke-like ST theories with scalar/matter coupling [14].

One of the simplest model of such a coupling is given by a field that couples universally to all the material field through a function in factor of the material part of the Lagrangian ¹. Such a field has been dubbed *chameleon* by [15, 16], even though the chameleon field originally refers to the *thin shell mechanism* [17] that appears in some models where the scalar/matter coupling appears through conformal factors in front of the metric entering in the material part of the Lagrangian [17–19]. Since it is easily seen that there is no conformal transformation that goes from the universal scalar/mater coupling action to the chameleon action, and since there is no proof we are aware of that the scalar-field we consider would generically behave as a *chameleon*, we won’t dubb our scalar field *chameleon*.

¹ Note that it differs from Damour and Polyakov toy model [7] by the fact that the universal coupling function is a factor of the material part of the Lagrangian only.

Especially, the Post-Newtonian (PN) limit of the Brans-Dicke theory for a universal scalar/matter coupling has been worked out by Saaïdi et al. [16] in the case of an inverse power law potential. In a recent comment [20], we argued about several mistakes made in Saaïdi et al. paper. In the current paper, we derive what we think is the actual post-Newtonian behavior of ST theories with universal scalar/matter coupling.

In section II, we derive the equations of motion coming from the considered action. Then, in section III we study the first Post-Newtonian perturbative order of such theories. Section IV then gives the trivial generalization to the full Scalar-Tensor case. Finally, section V is about some possible links of the theory considered here to extra-dimensional theories.

Throughout the paper, the notation conventions of the International Astronomical Union are used [21].

II. EQUATIONS OF MOTION

The action describing Brans-Dicke theory with a universal scalar/matter coupling can be written as follows:

$$S = \int d^4x \sqrt{-g} \left(\Phi R - \frac{\omega}{\Phi} (\partial_\sigma \Phi)^2 - V(\Phi) + 2f(\Phi) \mathcal{L}_m(g_{\mu\nu}, \Psi) \right), \quad (1)$$

where g is the metric determinant, R is the Ricci scalar constructed from the *physical* metric $g_{\mu\nu}$ ², \mathcal{L}_m is the material Lagrangian, Ψ represents the non-gravitational fields and $V(\Phi) \propto \Phi^{-n}$, with $n \in \mathbb{R}$. From this action, and defining

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g} \mathcal{L}_m)}{\delta g^{\mu\nu}}, \quad (2)$$

one gets the following equations of motion:

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R &= \frac{f(\Phi)}{\Phi} T_{\mu\nu} + \frac{\omega}{\Phi^2} (\partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2} g_{\mu\nu} (\partial_\alpha \Phi)^2) \\ &\quad + \frac{1}{\Phi} [\nabla_\mu \nabla_\nu - g_{\mu\nu} \square] \Phi - g_{\mu\nu} \frac{V(\Phi)}{2\Phi}, \end{aligned} \quad (3)$$

and

$$\frac{2\omega + 3}{\Phi} \square \Phi = \frac{f(\Phi)}{\Phi} T - 2f_{,\Phi}(\Phi) \mathcal{L}_m + V_{,\Phi}(\Phi) - 2 \frac{V(\Phi)}{\Phi}. \quad (4)$$

III. THE POST-NEWTONIAN DEVELOPMENT

Lets write the perturbations of the fields as follow:

$$\Phi = \Phi_0 + c^{-2} \varphi \quad (5)$$

$$g_{\mu\nu} = \eta_{\mu\nu} + c^{-2} h_{\mu\nu} + O(c^{-3}), \quad (6)$$

where $\eta_{\mu\nu}$ is the metric of Minkowski and Φ_0 is constant background field. It has to be stressed out that equation (6) is always valid in a local enough region of space-time³. Therefore, we are choosing not to restrict ourselves to the asymptotic flatness case. However, observations tell us that $g_{00}g_{ij} = -\delta_{ij}$ + unobserved quantities (so far), where $g_{00} \sim -1 + c^{-2} \{2G_{eff}M/r\} + O(c^{-4})$. Therefore, because of equations (3-4), the quantities $V(\Phi_0)/\Phi_0$ and $V_{,\Phi}|_{\Phi_0}$ are at best of order $8\pi G_{eff}T^{00}/c^4$, where G_{eff} is the effective (or local) gravitational constant as measured at low redshifts. In other words, the potential terms are at best first order terms; while Saaïdi et al. assumed that they were zeroth order terms in [16]⁴. Also, because we don't expect the potentials to be related to the particular solar system's energy density, it would be a suspicious coincidence that those terms would be precisely of order $8\pi G_{eff}T^{00}/c^4$ and

² In our model, we assume that $g_{\mu\nu}$ is the *physical* metric, in the sens that it is the one that describes actual time and length as measured by clocks and rods in our experiments [1].

³ The size L of such a region can be linked in some situations to the effective cosmological constant λ_{eff} through $L \ll \sqrt{1/\lambda_{eff}}$ [22].

⁴ As a result, they obtained an asymptotically flat solution that is however incompatible with their field equation where $V(\Phi)$ and $V_{,\Phi}(\Phi)$ wouldn't be asymptotically null.

not arbitrarily bellow. Indeed, a priori they could have any value that is not constrained by experiments (ie. roughly speaking, any value far less than $8\pi G_{eff} T^{00}/c^4 \sim 10^{-23} m^{-2}$ for Solar system objects). Therefore when dealing with Scalar-Tensor theories, while potentials can play an important role at cosmological scales, one either not considers potential at the Solar System scale [1, 2, 23, 24]; or considers only fine-tuned potentials where $V(\Phi_0) = V_{,\Phi}(\Phi_0) = 0$, while $V_{,,\Phi}(\Phi_0) = m^2$ [25–27])⁵. Indeed, if $V(\Phi_0) \neq 0$, equation (3) implies

$$\frac{V(\Phi_0)}{\Phi_0} = O(\lambda_{eff}), \quad (7)$$

where λ_{eff} is the effective (or local) cosmological constant as measured for "low-enough" redshifts. Observations tell us that $\lambda_{eff} \sim 10^{-52} m^{-2}$ while, for Solar system objects, one has $8\pi G_{eff} T^{00}/c^4 \sim 10^{-23} m^{-2}$. Besides, one has

$$V_{,\Phi}(\Phi_0) = -n \frac{V(\Phi_0)}{\Phi_0} + O(c^{-2} n \lambda_{eff}). \quad (8)$$

Moreover, equation (3) leads to $f(\Phi_0)/\Phi_0 \sim 8\pi G_{eff}/c^4$. Hence, unless $|n| \gtrsim 10^{29}$, $V_{,\Phi}$ can also be considered as a negligible quantity at the first perturbative order. (Of course, the discussion actually depends on the scale considered, as one expects from the relaxation of the asymptotic flatness assumption – see appendix A for a more detailed discussion). Hence, one writes

$$V_{,\Phi}(\Phi_0) = O(n \lambda_{eff}). \quad (9)$$

Now, assuming that at the lowest perturbative order one has $\mathcal{L}_m = -c^2 \rho + O(c^0) = T + O(c^0)$ [28–32], equations (3) and (4) can be re-written at the first perturbative order as the following:

$$R^{\mu\nu} = \frac{f(\Phi_0)}{\Phi_0} \left(T^{\mu\nu} - \frac{1}{2} g^{\mu\nu} T \right) + \frac{1}{\Phi_0} \left(\partial^\mu \partial^\nu + \frac{1}{2} g^{\mu\nu} \Delta \right) \Phi + O(c^{-4}, \lambda_{eff}), \quad (10)$$

$$\frac{2\omega + 3}{\Phi_0} \square \Phi = (1 + \Delta) \frac{f(\Phi_0)}{\Phi_0} T + O(c^{-4}, \lambda_{eff}, n \lambda_{eff}), \quad (11)$$

where

$$\Delta \equiv -\zeta \Phi_0 \ln_{,\Phi}(f(\Phi))|_{\Phi_0}, \quad (12)$$

ζ being equal to 2 in our case. (The case treated by [16] would correspond to $\zeta = 1/2$). From there, we can notice three possible cases :

- $\left| \frac{1+\Delta}{2\omega+3} \right| \gtrsim 1/2$: where the main source of the curvature is the second part of the rhs. of (10), which contains the scalar field terms. Therefore we dubb this case, the *strong coupling* regime.
- $\Delta = -1 (\Rightarrow \epsilon = 0) \Rightarrow f(\Phi) = \alpha \Phi^{1/\zeta} + n(\Phi)$, where α is a constant, $n(\Phi_0)$ and $n_{,\Phi}(\Phi_0)$ are small enough to be neglected in the field equations at the first perturbative order and $\zeta \neq 0$. In that case, the scalar-field is not coupled to matter at the first order and the PN limit of GR is recovered. Otherwise, one should notice that for $f(\Phi) = \alpha \Phi^{1/\zeta}$, GR equations are totally recovered for the field solution $\Phi = \Phi_0$.
- $0 < \left| \frac{1+\Delta}{2\omega+3} \right| < 1/2$: where the main source of the curvature is the first part of the rhs. of (10). Therefore we dubb this case, the *weak coupling* regime.

As one shall notice later on, it turns out that if one wants to have $\gamma \sim 1$, one needs to either be in the weak coupling regime or in the non-coupling one.

Since we are interested in the γ parameter only, and in order to simplify the discussion, we won't discuss the c^{-3}

⁵ Note that a priori, there is no reason why – following $V(\Phi_0)$ and $V_{,\Phi}(\Phi_0) - V_{,,\Phi}(\Phi_0)$ shouldn't be arbitrarily small. Therefore, in the following we will refer to theories with $V_{,,\Phi}(\Phi_0) = m^2$ as *locally massive ST*.

metric term. Therefore, we re-write the field equations at the 1-PN/RM level ⁶:

$$R^{\mu\nu} = \frac{f(\Phi_0)}{\Phi_0} \left(T^{\mu\nu} - \frac{1}{2} g^{\mu\nu} T \right) + \frac{c^{-2}}{\Phi_0} \left(\partial^\mu \partial^\nu + \frac{1}{2} g^{\mu\nu} \Delta \right) \varphi + O(c^{-3}, \lambda_{eff}, n \lambda_{eff}), \quad (13)$$

$$\frac{2\omega + 3}{\Phi_0} \Delta \Phi = (1 + \Delta) \frac{f(\Phi_0)}{\Phi_0} T + O(c^{-3}, \lambda_{eff}, n \lambda_{eff}). \quad (14)$$

(In the following, for simplicity we shall write $O(\lambda_{eff}, n \lambda_{eff}) = O(\lambda_{eff})$.) These equations correspond to the equations of a non-massive BD theory – with universal scalar/matter coupling – at the first PN/RM order. Defining

$$\sigma \equiv T^{00}/c^2 + O(c^{-2}), \quad (15)$$

$$G_{eff} \equiv \left(1 + \frac{1 + \Delta}{2\omega + 3} \right) \frac{c^4}{8\pi} \frac{f(\Phi_0)}{\Phi_0}, \quad (16)$$

$$\gamma \equiv \frac{2\omega + 2 - \Delta}{2\omega + 4 + \Delta}, \quad (17)$$

the previous equations can be re-written as follows:

$$R^{00} = c^{-2} \{ 4\pi G_{eff} \sigma \} + O(c^{-3}, \lambda_{eff}), \quad (18)$$

$$R^{ij} = c^{-2} \left\{ -\delta_{ij} \gamma 4\pi G_{eff} \sigma + \frac{1}{\Phi_0} \partial_i \partial_j \varphi \right\} + O(c^{-3}, \lambda_{eff}) \quad (19)$$

$$\frac{1}{\Phi_0} \Delta \varphi = -\frac{2 + 2\Delta}{2\omega + 4 + \Delta} 4\pi G_{eff} \sigma + O(c^{-1}, c^2 \lambda_{eff}). \quad (20)$$

It is then straightforward to show that the metric solution can be put under the following standard PN/RM form

$$g_{00} = -1 + c^{-2} \frac{2w}{c^2} + O(c^{-3}, \lambda_{eff}), \quad (21)$$

$$g_{0i} = O(c^{-3}, \lambda_{eff}), \quad (22)$$

$$g_{ij} = \delta_{ij} \left(1 + c^{-2} \frac{2\gamma w}{c^2} \right) + O(c^{-3}, \lambda_{eff}), \quad (23)$$

where γ is indeed a constant given by (17), and where w satisfies the equation of Newton at the first perturbative order :

$$\Delta w = -4\pi G_{eff} \sigma + O(c^{-1}, c^2 \lambda_{eff}). \quad (24)$$

The important fact to notice is that, depending on the value of Δ (and thus depending on the coupling function), $1 - \gamma$ could be either positive, null ⁷ or negative; while STT without coupling predict a positive value for finite value of ω .

A. Remark on the current constraints coming from propagation of light observations

As shown in appendix B, the geometric optic limit of the modified Maxwell equations leads to the usual geodesic equation for the propagation of light. Hence, from (17), one gets that the weak coupling is constrained by observations involving propagation of light as follows:

$$\left| \frac{1 + \Delta}{2\omega + 3} \right| \lesssim \frac{|1 - \gamma_{obs}|}{2}, \quad (25)$$

where γ_{obs} is the value given by current observational constraints on the PN parameter γ .

⁶ PN/RM stands for Post Newtonian/Relativistic Motion. It means that the development of the Post-Newtonian metric is developed to the order that has to be taken into account when dealing with test particles with relativistic velocities only. On the contrary, PN/SM stands for Post-Newtonian/Slow Motion. It means that the development of the Post-Newtonian metric is developed to the order that has to be taken into account when dealing with test particles with non-relativistic velocities only [23, 33].

⁷ The author has recently became aware of the paper of Moffat and Toth [34] in which they explored such a possibility in order to argue the possible solar system viability of Modified Gravity Theory (MOG) [35, 36].

IV. GENERALIZATION TO THE SCALAR-TENSOR CASE

The generalization of the action (1) to the general scalar-tensor case results in the following modifications of the field equations (3)-(4)

$$\begin{aligned} & \Phi(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R) \\ &= f(\Phi)T_{\mu\nu} + \frac{\omega(\Phi)}{\Phi}(\partial_\mu\Phi\partial_\nu\Phi - \frac{1}{2}g_{\mu\nu}(\partial_\sigma\Phi)^2) \\ &+ [\nabla_\mu\nabla_\nu - g_{\mu\nu}\square]\Phi - g_{\mu\nu}\frac{V(\Phi)}{2}, \end{aligned} \quad (26)$$

and

$$\begin{aligned} (2\omega(\Phi) + 3)\square\Phi &= f(\Phi)T - 2\Phi f_{,\Phi}(\Phi)\mathcal{L}_m \\ &+ (\Phi V_{,\Phi}(\Phi) - 2V(\Phi)) - \omega_{,\Phi}(\partial_\sigma\Phi)^2. \end{aligned} \quad (27)$$

The 1-PN/RM limit of equations (26-27) is the same than for the simple BD case considered previously. It is not surprising that the field equations at the 1-PN/RM order are the same in both the Scalar-Tensor and Brans-Dicke cases. Indeed, the changes from equations (3)-(4) and (26)-(27) arise in factor of $(\partial_\sigma\Phi)^2$ -like terms that are of order $O(c^{-4})$. Thus, those terms will impact the calculation at the 1PN/SM or 2PN/RM level only [23].

Therefore, our result (21)-(24) holds for the Scalar-Tensor case (with ω replaced by $\omega(\Phi_0)$).

Otherwise, using the Einstein representation gives the same results as one can see in appendix D.

V. REMARK REGARDING THEORIES WITH EXTRA DIMENSIONS

In most cases, compactified extra dimensions imply ω to be of the order of -1 [5, 37]. Hence, in order to pass solar system tests that give $\omega \gtrsim 10^4$ for non-massive scalar-fields [26], extra-dimensional theories have to imply a sufficient mass for the effective scalar field. Indeed, a very massive scalar-field has a frozen spatial dynamics on solar system scales, such that every value of ω satisfy solar system observations [26] ($|1 - \gamma| \lesssim 10^{-4}$ [38]). However, in [39] is described a mechanism allowing to get rid of this condition. But as we can see, the universal scalar/matter coupling allows to do the same. Indeed, as long as $\Delta = -1 + |\mu|$, with μ small enough, the theory could pass the γ -parameter solar system tests, no matter the value of ω . However, at the phenomenological level considered here, it requires a fine-tuning of the theory. Besides, it is worth stressing that those theories are also constrained by the tests of the equivalence principle. For instance, among other violations of the Equivalence Principle – such as the variation of the fundamental *constants* – there is an energy transfer between material and scalar fields such that the conservation equation writes:

$$\nabla_\sigma [f(\Phi)T^{\mu\sigma}] = \mathcal{L}_m f_{,\phi}(\Phi)\partial^\mu\Phi. \quad (28)$$

However, such an energy transfer can actually have some virtue, as shown for general material to scalar-field energy transfers in [40] or more specifically for the theory considered here in [15]. Besides, one could invoke the so-called *Least Coupling Principle* in order to explain the smallness of the current coupling [7, 8]. Indeed, it has been argued that in some cases (see the following subsection as well as appendix C), the scalar-field is naturally driven toward a decoupling to matter through the cosmic evolution. In that case, one could have $f_{,\phi}(\Phi)|_{\Phi_0} \sim 0$; while Δ could still be finite (see next subsection).

A. Example: Damour&Polyakov Toy Model

From the observation in [8] that a kind of universality between the coupling functions leads to a strong attracting mechanism – realising the so-called *Principle of Least Coupling* – Damour and Polyakov proposed a toy model in [7] – suggested by an effective string massless modes action in String theories – with one universal coupling function $B(\Psi)$:

$$S = \int d^4x \sqrt{-g} B(\Psi) \left[R + 4\square\Psi - 4(\partial_\sigma\Psi)^2 + \alpha\mathcal{L}_m \right]. \quad (29)$$

Admitting that $B(\Psi) \equiv \Phi$ is invertible (at least locally) and that the inversion around Ψ_0 writes $\Psi = A(\Phi)$ – along with the fact that divergences don’t contribute to the equations of motion – one shows that the previous action reduces to ⁸:

$$S = \int d^4x \sqrt{-g} \left(\Phi R - \frac{\omega(\Phi)}{\Phi} (\partial_\sigma \Phi)^2 + \alpha \Phi \mathcal{L}_m \right), \quad (30)$$

with

$$\omega(\Phi) \equiv 4\Phi A_{,\Phi}(\Phi) [1 + A_{,\Phi}(\Phi)]. \quad (31)$$

It means that their toy model locally corresponds to $f(\Phi) = \alpha\Phi/2$. Now, if one takes the case where $\zeta = 2$ – as advocates by section II – then one finds $\Delta = -2$. Thus, thanks to equation (17), one deduces $\gamma > 1$. (It is important to notice that one finds $\gamma > 1$ as well when considering their original action without the inversion considered here. A generalization to any parametrization can be found in the appendix C.).

VI. CONCLUSION

First, it is shown in this paper that Brans-Dicke theories with a power law potential and a universal scalar/matter coupling reduces to the usual standard PN metric at the 1PN/RM order, as soon as one takes actual numerical values into account. Moreover, it is emphasized that the result holds when the BD theory is generalized to the ST theory. In both cases, the only difference regarding the metric with the case considered in Will’s book for instance [2], is that the coefficient γ is modified because of the coupling of the scalar field with matter:

$$\gamma \equiv \frac{2\omega + 2 - \Delta}{2\omega + 4 + \Delta}, \quad (32)$$

$$\Delta \equiv -\zeta \Phi_0 \ln_{,\Phi} f(\Phi)|_{\Phi_0} \quad (33)$$

However, conversely to STT without coupling, it is shown that the $1 - \gamma$ can be either positive, null or negative, depending on the coupling function $f(\Phi)$. The knowledge of this behavior is crucial in order not to wrongly rule out ST theories if the post-Newtonian parameter γ was measured to be more than one.

Second, it is argued that such a scalar-field, if generated by compactification of extra dimensions, could allow theories with compactified extra dimensions to pass solar system tests, even if they don’t lead to a mass for the scalar field in the reduced 4-dimension space-time.

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Appendix A: λ_{eff} and the scale of its influence

While the matter source term in the field equations is bounded, the " λ " source terms are not and hence, their influence grow with the increasing of the scale considered. Mathematically speaking, it means that while the former leads to "external" spherical harmonic solutions (ie. $w \propto 1/r^L$), the latter leads to "internal" spherical harmonic solutions (ie. $w \propto r^L$). Thus, the discussion about orders of magnitude in section III has to be specified – although the huge difference between the value of $4\pi G_{eff} T^{00}/c^4$ and λ_{eff} don't leave many doubts regarding the smallness of the λ contribution at the PN level.

The 1-PN/RM field equations for which the potential and its derivative have not been neglected write:

$$c^2 R^{00} = \frac{c^4 f(\Phi_0)}{2\Phi_0} \sigma - \frac{\Delta \varphi}{2\Phi_0} - c^2 \frac{V(\Phi_0)}{2\Phi_0} + O(c^{-2}), \quad (A1)$$

$$\begin{aligned} \frac{\Delta \varphi}{\Phi_0} = & \frac{1 + \Delta}{2\omega + 3} \frac{f(\Phi_0)}{\Phi_0} \sigma + \frac{c^2}{2\omega + 3} \left\{ V_{,\Phi}(\Phi_0) - \frac{2V(\Phi_0)}{\Phi_0} \right\} \\ & + O(c^{-2}). \end{aligned} \quad (A2)$$

Combining those two equations, one gets the following solution for the gravitational potential at the Newtonian level:

$$w = G_{eff} \int \frac{\sigma(t, \vec{x}')}{|\vec{x} - \vec{x}'|} d^3 x' + \frac{\lambda^*}{6} r^2 + O(c^{-2}), \quad (\text{A3})$$

where

$$\lambda^* \equiv c^2 \left\{ \frac{2\omega + 1}{4\omega + 6} \frac{V(\Phi_0)}{\Phi_0} + \frac{V_{,\Phi}(\Phi_0)}{4\omega + 6} \right\}. \quad (\text{A4})$$

One can note that this simple result 1/ coincides with GR case – with a non-null cosmological constant – at the Newtonian order (with λ replaced by λ^*), 2/ is quite different from the result obtained by [16] and 3/ is in accordance with the relaxation of the asymptotic flatness assumption in section III.

Since we saw that in this class of model, ω is not trivially constrained by observations, the second term of the r.h.s. is not necessarily small compared to the first term, and thus should not be neglected a priori. However, under the class of potentials considered here, and assuming that $|n|$ is small enough, we shall consider that $V_{,\Phi}(\Phi_0) = O(V(\Phi_0)/\Phi_0)$, such that we indeed have $c^{-2}\lambda^* = O(\lambda_{eff}) \sim 10^{-52} m^{-2}$. Under those assumptions, if we require that the λ -term stays N orders of magnitude below the *Newtonian-term*, it leads to the following requirement:

$$r < \left(\frac{6G_{eff}M}{10^N c^2 \lambda_{eff}} \right)^{1/3}. \quad (\text{A5})$$

For the solar system, a safety of 6 orders of magnitude implies that the scale of the experiment has to be less than 1 parsec (~ 3 light-years). From this, we conclude that it is safe enough to neglect the potential in the field equations, if the potential is linked to the value of the locally measured cosmological constant as discussed in section III.

Also, one should notice that the consequences of a gravitational potential with a diverging r^2 -term as in (A3) have been thoroughly studied and compared to data (see for instance [41] and references therein).

Appendix B: The Geometric Optic limit

Lets consider the geometric optic limit of Electromagnetism in the theory studied here. The electromagnetic field amplitude considered being extremely weak – a laser of a few Watts for instance – the electromagnetic field won't affect the field equations previously considered (ie. photons are considered as test particles). However, the Maxwell equation in vacuum is modified by the scalar field in the following way:

$$\nabla_\sigma (f(\Phi) F^{\mu\sigma}) = 0. \quad (\text{B1})$$

Using the Lorenz Gauge ($\nabla_\sigma A^\sigma = 0$), along with equation (20), one puts this equation under the 1-PN/RM following form:

$$-\square A^\mu + g^{\mu\epsilon} R_{\gamma\epsilon} A^\gamma + \varkappa (\nabla^\mu A^\sigma - \nabla^\sigma A^\mu) \partial_\sigma w = O(c^{-3}), \quad (\text{B2})$$

where

$$\varkappa \equiv c^{-2} \Phi_0 \frac{f_{,\Phi}}{f} \Big|_{\Phi_0} \frac{2 + 2\Delta}{2\omega + 4 + \Delta}. \quad (\text{B3})$$

Following the analysis made in [42], we expand the 4-vector potential as follows:

$$A^\mu = \Re \left\{ (a^\mu + \epsilon b^\mu + O(\epsilon^2)) \exp^{i\theta/\epsilon} \right\}, \quad (\text{B4})$$

The two first leading orders of equation (B2) respectively give:

$$k_\sigma k^\sigma = 0, \quad (\text{B5})$$

where $k_\sigma \equiv \partial_\sigma \theta$, and

$$a^\mu \nabla_\sigma k^\sigma + 2k^\sigma \nabla_\sigma a^\mu + \varkappa (k^\mu a^\sigma - k^\sigma a^\mu) \partial_\sigma w = 0. \quad (\text{B6})$$

Remembering that the Lorenz Gauge condition gives $k_\sigma a^\sigma = 0$ at the leading order, one gets:

$$k^\sigma \nabla_\sigma k^\mu = 0. \quad (\text{B7})$$

This equation is the usual geodesic equation, showing that the presence of the scalar-field won't affect light ray trajectories at the geometric optic approximation. However, defining $a^\mu = af^\mu$, the propagation equation for the scalar amplitude (a) as well as the propagation equation for the polarization vector (f^μ) are modified:

$$k^\sigma \nabla_\sigma a = -\frac{a}{2} \nabla_\sigma k^\sigma + \frac{\varkappa}{2} a k^\sigma \partial_\sigma w, \quad (\text{B8})$$

$$k^\sigma \nabla_\sigma f^\mu = -\frac{\varkappa}{2} k^\mu f^\sigma \partial_\sigma w. \quad (\text{B9})$$

From there follows that the conservation law of "photon number" is modified:

$$\nabla_\sigma (k^\sigma a^2) = \varkappa a^2 k^\sigma \partial_\sigma w. \quad (\text{B10})$$

One notes that the last three equations give alternative ways to put constraints on those theories. Those ways should be investigated using the relevant literature.

Otherwise, one should notice that $\Delta = -1$ implies $\varkappa = 0$ in addition to $\gamma = 1$.

Appendix C: General parametrization of the scalar field

If, instead of action (1), one starts with the following general action:

$$S = \int d^4x \sqrt{-g} \left\{ F(\Phi) R - Z(\Phi) (\partial_\sigma \Phi)^2 + f(\Phi) \mathcal{L}_m(g_{\mu\nu}, \Psi) \right\}, \quad (\text{C1})$$

then the γ parameter re-writes:

$$\gamma = \frac{2ZF + (2 - \Delta)(F_{,\Phi})^2}{2ZF + (4 + \Delta)(F_{,\Phi})^2} \Big|_{\Phi_0}, \quad (\text{C2})$$

with

$$\Delta \equiv -2 \frac{F}{F_{,\Phi}} \Big|_{\Phi_0} \frac{f_{,\Phi}}{f} \Big|_{\Phi_0}. \quad (\text{C3})$$

One can note that, as emphasized in section V A, having $F = f$ as in the Damour and Polyakov's case implies $\Delta = -2$ and therefore $\gamma > 1$.

Otherwise, let us remark that $f = \sqrt{F}$ leads to $\gamma = 1$ (as well as $\varkappa = 0$, cf. appendix B) regardless the value of the kinetic function Z . Besides, one should notice that f and F would then share the same local minimums. Hence, according to Damour and Polyakov's study [8], such a theory is driven toward a weak scalar/mater coupling through cosmic evolution. Therefore, such theories (ie. for any Z , and any F with a local minimum Φ_m) can be in perfect accordance with the solar system phenomenology, as it is currently known.

Appendix D: Using the Einstein representation

The result presented in this paper does not depend on the representation used to do the calculations as long as one has fixed the *physical* metric – ie. as long as one has explicitly fixed in the theory, which metric characterizes the proper time (τ) in actual experiments ($d\tau^2 = g_{\alpha\beta} dx^\alpha dx^\beta$, where $g_{\alpha\beta}$ is the so-called *physical* metric) [1]. However, it is always interesting to re-derive the calculations in the Einstein representation in order to check the results obtained while using the Jordan representation only. Let's consider the massless case, the action writes in the Jordan and Einstein representation respectively as follows:

$$S = \int d^4x \sqrt{-g} \left(\Phi R - \frac{\omega(\Phi)}{\Phi} g^{\alpha\beta} \partial_\alpha \Phi \partial_\beta \Phi \right) + S_m, \quad (\text{D1})$$

$$= \int d^4x \sqrt{-\tilde{g}} \left(\tilde{R} - \left(\omega(\Phi(\varphi)) + \frac{3}{2} \right) \tilde{g}^{\alpha\beta} \partial_\alpha \varphi \partial_\beta \varphi \right) + S_m, \quad (\text{D2})$$

where $g^{\alpha\beta} \equiv \Phi \tilde{g}^{\alpha\beta}$, $\sqrt{-g} = \Phi^{-2} \sqrt{-\tilde{g}}$ and $\varphi \equiv \ln \Phi$. By definition, the material part of the action (S_m) writes:

$$S_m = \int d^4x \sqrt{-g} \, 2f(\Phi) \mathcal{L}_m, \quad (\text{D3})$$

$$= \int d^4x \sqrt{-\tilde{g}} \, 2f(\Phi(\varphi)) \tilde{\mathcal{L}}_m. \quad (\text{D4})$$

Therefore, by definition, one has $\tilde{\mathcal{L}}_m = \Phi^{-2} \mathcal{L}_m$. The variation of equation (D3) for relevant fields leads to:

$$\delta S_m = \int d^4x \sqrt{-g} \left(-f(\Phi) T_{\alpha\beta} \delta g^{\alpha\beta} + 2f_{,\Phi}(\Phi) \mathcal{L}_m \delta\Phi \right). \quad (\text{D5})$$

Now, since one has $g^{\alpha\beta} \equiv \Phi \tilde{g}^{\alpha\beta}$, the variation of the *physical* metric gives

$$\delta g^{\alpha\beta} = \tilde{g}^{\alpha\beta} \delta\Phi + \Phi \delta\tilde{g}^{\alpha\beta}. \quad (\text{D6})$$

Therefore, equation (D5) writes:

$$\begin{aligned} \delta S_m = & \int d^4x \sqrt{-g} \left(-\Phi f(\Phi) T_{\alpha\beta} \delta\tilde{g}^{\alpha\beta} \right. \\ & \left. + [-f(\Phi) \tilde{g}^{\alpha\beta} T_{\alpha\beta} + 2f_{,\Phi}(\Phi) \mathcal{L}_m] \delta\Phi \right). \end{aligned} \quad (\text{D7})$$

Now, using $T_{\alpha\beta} = \Phi \tilde{T}_{\alpha\beta}$, $\tilde{T} \equiv \tilde{g}^{\alpha\beta} \tilde{T}_{\alpha\beta}$, $\delta\varphi = \delta\Phi/\Phi$ and $\Phi f_{,\Phi}(\Phi) = f_{,\varphi}(\Phi(\varphi))$, one gets:

$$\begin{aligned} \delta S_m = & \int d^4x \sqrt{-\tilde{g}} \left(-f(\Phi(\varphi)) \tilde{T}_{\alpha\beta} \delta\tilde{g}^{\alpha\beta} \right. \\ & \left. - \left[1 - 2 \frac{f_{,\varphi}(\Phi(\varphi))}{f(\Phi(\varphi))} \frac{\tilde{\mathcal{L}}_m}{\tilde{T}} \right] f(\Phi(\varphi)) \tilde{T} \delta\varphi \right). \end{aligned} \quad (\text{D8})$$

However, since $\tilde{\mathcal{L}}_m = \Phi^{-2} \mathcal{L}_m$ and $\tilde{T} = \Phi^{-2} T$, then $\tilde{\mathcal{L}}_m/\tilde{T}$ reduces to $\mathcal{L}_m/T = 1 + O(c^{-2})$ [32]. Therefore, for $\Delta = -1$, one derives that $\varphi = C + O(c^{-4})$, where C is an arbitrary constant. But in the Einstein representation, the metric satisfies the Strong Spatial Isotropy Condition [43] (ie. $\gamma = 1$ for the metric in the Einstein representation). Indeed, the space-space part of Einstein tensor in the Einstein representation is of order c^{-4} ($\tilde{R}^{ij} - 1/2 \tilde{g}^{ij} \tilde{R} = O(c^{-4})$) as one can easily check from the field equations obtained from the action (D2) (see discussions in [23] and [43]).

Therefore, when $\Delta = -1$, the conformal transformation $g_{\alpha\beta} = e^{-\varphi} \tilde{g}_{\alpha\beta} = e^{-C} \tilde{g}_{\alpha\beta} + O(c^{-4})$ corresponds to a simple redefinition of the metric's units at the 1.5PN/RM level, and one has $\gamma = 1$ in the metric of the Jordan representation as well.

In conclusion, one indeed gets the same result whether one derives everything in the Jordan representation or uses the Einstein representation as an intermediate step.